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On the exact evaluation of the membrane instanton superpotential in M-theory on G_2-holonomy manifolds

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Abstract: Following the work of [1] on the exact evaluation of the non-perturbative contribution to the superpotential from open-membrane instanton in Heterotic M-theory, we evaluate systematically the contribution to the superpotential of a membrane instanton obtained by wrapping of a single M2-brane, once, on an isolated supersymmetric 3-cycle in a G_2-holonomy manifold. We then try to relate the results obtained to those sketched out in [2]. We also do a heat-kernel asymptotics analysis to see whether one gets similar UV-divergent terms for (one or both of) the bosonic and fermionic determinants indicative of (partial) cancellation among them. The answer is in the affirmative, as expected by the supersymmetry of the starting membrane action. This work is a first step, both, in extending the work of [2] to the evaluation of non-perturbative superpotentials for non-rigid supersymmetric 3-cycle wrappings, and in understanding the large-N Chern-Simons/closed type-A topological string theory duality of [3] from M-theory point of view.

Keywords: Non-perturbative Effects, M-Theory.

Contents

1. Introduction

String and M-theories on manifolds with G_2 and Spin(7) holonomies have become an active area of research, after construction of explicit examples of such manifolds by Joyce [5]. Some explicit metrics of non-compact manifolds with the above-mentioned exceptional holonomy groups have been constructed by Brandhuber et al. [6] and Cvectic et al. [7].

Gopakumar and Vafa in [3], had conjectured that similar to the large-N Chern-Simons/open topological string theory duality of Witten, large-N Chern-Simons on S^3 is dual to closed type-A topological string theory on an S^2-resolved conifold geometry. This conjecture was verified for arbitrary genus g and arbitrary t'Hooft coupling. This duality was embedded by Vafa in type IIA to conjecture the following duality: type IIA with N D6-branes wrapping the S^3 in the S^3 resolved conifold, is dual to type IIA with the D6-branes being replaced by RR flux through $O(-1)+O(-1)$ line bundle over $\mathbf{CP^1}$. The latter (duality) was proven by uplifting it to M-theory on a G_2 holonomy manifold by Atiyah, Maldacena and Vafa [4]. The G_2 holonomy manifold that was considered by Atiyah et al. in [4] was a spin bundle over S^3 with the topology of $R^4 \times S^3$. The volume of S^3, gets complexified to V_M with the imaginary part given by the flux of the 3-form of M-theory through S^3. Further, it is argued that M-theory on the above-mentioned G_2-holonomy manifold, modded out by a group G, with complex volume $-V_M$ is given by M-theory modded out by a group G', related by a $\mathbb{Z}_2$ outer automorphism to the group G, and complex volume V_M. Then, modding out the above-mentioned G_2-holonomy manifold by two $\mathbb{Z}_2$ actions, the two sides of Vafa's type-IIA dualities were obtained: the one with fixed points yielding type-IIA theory with D6-branes wrapping S^3, and the one without fixed points yielding type-IIA theory with RR flux through S^2. The M-theory lifts of both sides are related by a $V_M \leftrightarrow -V_M$ "flop"

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It will be interesting to be able to lift the above-mentioned Gopakumar-Vafa duality to M-theory on a G_2-holonomy manifold. As the type-A topological string theory's partition function receives contributions only from holomorphic maps from the world-sheet to the target space, and apart from constant maps, instantons fit the bill, as a first step we should look at obtaining the superpotential contribution of multiple wrappings of M2-branes on supersymmetric 3-cycles in a suitable G_2-holonomy manifold(membrane instantons). A sketch of the result anticipated for a single M2-brane wrapping an isolated supersymmetric 3-cycle, was given by Harvey-Moore. In this work, we have worked out the exact expression for the same, using techniques developed in [1] on evaluation of the non-perturbative contribution to the superpotential of open membrane instantons obtained by wrapping the M2-brane on an interval $[0,1]$ times (thus converting the problem to that of a heterotic string wrapping) a holomorphic curve in a Calabi-Yau three-fold.

The paper is organized as follows: in section 2, we spell out the details of the calculation and obtain the exact form of the non-perturbative contribution to the superpotential of a single M2-brane wrapping an isolated supersymmetric cycle embedded in the G_2-holonomy manifold. In section 3, we compare our result with the one sketched out in [2]. Section 4 has a discussion on heat kernel asymptotics related to looking for possible cancellations among the bosonic and fermionic determinants. Section 5 has the conclusion and some speculative remarks about connecting this M-theory result (including its possible extension to multiple-covering of wrapping of the M2-brane on supersymmetric cycles in the G_2-holonomy manifold) to the Schwinger 1-loop calculation and the large-N limit of the Chern-Simons partition function of [16] to get a step closer in uplifting the Gopakumar/Vafa duality to M-theory.

2. Evaluation of the membrane instanton contribution to the superpotential

As given in [2], the euclidean action for an M2-brane is given by the following Bergshoeff, Sezgin, Townsend action:

$$\mathcal{S}_\Sigma = \int d^3z \left[\frac{\sqrt{g}}{l_{11}^3} - \frac{i}{3!}\epsilon^{ijk}\partial_i\mathbb{Z}^M\partial_j\mathbb{Z}^N\partial_k\mathbb{Z}^P C_{MNP}(X(s),\Theta(s))\right], \tag{2.1}$$

where $\mathbb{Z}$ is the map of the M2-brane world-volume to the the $D=11$ target space M_{11}, both being regarded as supermanifolds. The g in (2.1), is defined as:

$$g_{ij} = \partial_i\mathbb{Z}^M\partial_j\mathbb{Z}^N\mathbf{E}_M^A\mathbf{E}_N^B\eta_{AB}\,, \tag{2.2}$$

where $\mathbf{E}_M^A$ is the supervielbein, given in [2]. $X(s)$ and $\Theta(s)$ are the bosonic and fermionic coordinates of $\mathbb{Z}$. After using the static gauge and κ-symmetry fixing, the physical degrees of freedom, are given by $y^{m''}$, the section of the normal bundle to the M2-brane world volume, and $\Theta(s)$, section of the spinor bundle tensor product: $S(T\Sigma)\otimes S^-(N)$, where the $-$ is the negative Spin(8) chirality, as under an orthogonal decomposition of $TM_{11}|_\Sigma$ in terms of tangent and normal bundles, the structure group Spin(11) decomposes into Spin(3) $\times$ Spin(8).

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The action in (2.1) needs to be expanded up to $O(\Theta^2)$, and the expression is (one has to be careful that in euclidean $D=11$, one does not have a Majorana-Weyl spinor or a Majorana spinor) given as:

$$\mathcal{S}_\Sigma = \int_\Sigma \Bigg[C + \frac{i}{l_{11}^3}\,\mathrm{vol}(h) + \frac{\sqrt{h_{ij}}}{l_{11}^3}\left(h^{ij}D_i y^{m''}D_j y^{n''}h_{m''n''} - y^{m''}\mathcal{U}_{m''n''}y^{n''} + O\left(y^3\right)\right) + \\ + \frac{i}{l_{11}^3}\sqrt{h_{ij}}\frac{1}{2}\left(\bar{\Psi}_M V^M - \bar{V}^M\Psi_M\right) + 2\frac{\sqrt{h_{ij}}}{l_{11}^3}h^{ij}\bar{\Theta}\Gamma_i D_j\Theta + O(\Theta^3)\Bigg], \tag{2.3}$$

where we follow the conventions of [2]: V_M being the gravitino vertex operator, Ψ being the gravitino field that enters via the supervielbein $\mathbf{E}_M^A$, $\mathcal{U}$ is a mass matrix defined in terms of the Riemann curvature tensor and the second fundamental form (see (A.4)).

After κ-symmetry fixing, like [2], we set $\Theta_2^{A\dot{a}}(s)$, i.e. the positive Spin(8)-chirality to zero, and following [1], will refer to $\Theta_1^{Aa}(s)$ as θ.

The Kaluza-Klein reduction of the $D=11$ gravitino is given by:

$$\begin{aligned} dx^M\Psi_M &= dx^\mu\Psi_\mu + dx^m\Psi_m\,, \\ \Psi_\mu(x,y) &= \psi_\mu(x)\otimes\vartheta(y)\,, \\ \Psi_m(x,y) &= l_{11}^3\sum_{I=1}^{b_3}\omega^{(3)}_{I,mnp}(y)\Gamma^{pq}\chi^I(x)\otimes\tilde{\eta}(y)\,, \end{aligned} \tag{2.4}$$

where we do not write the terms obtained by expanding in terms of $\{\omega^{(2)}_{I,mn}\}$, the harmonic 2-forms forming a basis for $H^2(X_{G_2},\mathbb{Z})$, as we will be interested in M2-branes wrapping supersymmetric 3-cycles in the G_2-holonomy manifold. For evaluating the non-perturbative contribution to the superpotential, following [2], we will evaluate the fermionic 2-point function: $\langle\chi^i(x_1^u)\chi^j(x_2^u)\rangle$ (where $x_{1,2}$ are the $\mathbb{R}^4$ coordinates and u [and later also v]$\equiv 7,8,9,10$ is [are] used to index these coordinates), and drop the interaction terms in the $D=4,\mathcal{N}=1$ supergravity action. The corresponding mass term in the supergravity action appears as $\partial_i\partial_j W$, where the derivatives are evaluated w.r.t. the complex scalar obtained by the Kaluza-Klein reduction of $C+\frac{i}{l_{11}^3}\Phi$ using harmonic three forms forming a basis for $H^3(X_{G_2},\mathbb{R})$. One then integrates twice to get the expression for the superpotential from the 2-point function.

The bosonic zero modes are the four bosonic coordinates that specify the position of the supersymmetric 3-cycle, and will be denoted by $x_0^{7,8,9,10}\equiv x_0^u$. The fermionic zero modes come from the fact that for every θ_0 that is the solution to the fermionic equation of motion, one can always shift θ_0 to $\theta_0+\theta'$, where $D_i\theta'=0$. This $\theta'=\vartheta\otimes\eta$, where ϑ is a $D=4$ Weyl spinor, and η is a covariantly constant spinor on the G_2-holonomy manifold.

After expanding the M2-brane action in fluctuations about solutions to the bosonic and fermionic equations of motion, one gets that:

$$\mathcal{S}|_\Sigma = \mathcal{S}_0^y + \mathcal{S}_0^\theta + \mathcal{S}_2^y + \mathcal{S}_2^\theta\,, \tag{2.5}$$

where

$$\mathcal{S}_0^y \equiv \mathcal{S}_\Sigma|_{y_0,\theta_0}\,;$$

$$\mathcal{S}_0^\theta \equiv \mathcal{S}_\Sigma^\theta + \mathcal{S}_\Sigma^{\theta^2}|_{y_0,\theta_0}\,;$$
$$\mathcal{S}_2^y \equiv \frac{\delta^2 \mathcal{S}_\Sigma}{\delta y^2}|_{y_0,\theta_0=0}(\delta y)^2\,;$$
$$\mathcal{S}_2^\theta \equiv \frac{\delta^2 \mathcal{S}}{\delta\theta^2}|_{y_0,\theta_0=0}(\delta\theta)^2\,. \tag{2.6}$$

In (2.6), following [1], we consider classical values of coefficients of $(\delta y)^2, (\delta\theta)^2$ terms, as fluctuations are considered to be of $\mathcal{O}(\sqrt{\alpha'})$.

Now,

$$\langle \chi^i(x_1^u)\chi^j(x_2^u)\rangle = \int \mathcal{D}\chi e^{\text{K.E of }\chi}\,\chi^i(x)\chi^j(x)\int d^4x_0 e^{-\mathcal{S}_0^y}\times$$
$$\times\int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta}\int \mathcal{D}\delta y^{m''}e^{-\mathcal{S}_2^y}\int \mathcal{D}\delta\bar{\theta}\mathcal{D}\delta\theta e^{-\mathcal{S}_2^\theta}\,. \tag{2.7}$$

We now evaluate the various integrals that appear in (2.7) above starting with $\int d^4x e^{-\mathcal{S}_0^y}$:

$$\int d^4x_0 e^{-\mathcal{S}_0^y} = \int d^4x_0 e^{\left[iC-\frac{1}{l_{11}^3}\text{vol}(h)\right]}\,. \tag{2.8}$$

Using the eleven-dimensional euclidean representation of the gamma matrices as given in [2],

$$\Gamma^{1,2,3} = \sigma^{1,2,3}\otimes\begin{pmatrix}-\mathbf{1}_8 & 0\\ 0 & \mathbf{1}_8\end{pmatrix},$$
$$\Gamma^{4,5,6,7,8,9,10} = \mathbf{1}_2\otimes\begin{pmatrix}0 & \gamma^{1,2,3,4,5,6,7}\\ -\gamma^{1,2,3,4,5,6,7} & 0\end{pmatrix},$$
$$\Gamma^{11} = \mathbf{1}_2\otimes\begin{pmatrix}0 & \gamma^8\\ \gamma^8 & 0\end{pmatrix}, \tag{2.9}$$

where $\gamma^{1,2,3,4,\ldots,8}\in Cl(8)$. and that on-shell,

$$\mathcal{S}_0^\theta + \mathcal{S}_0^{\theta^2}|_\Sigma = \frac{i}{2l_{11}^3}\int_\Sigma \sqrt{h_{ij}}\,\bar{\Psi}_M V_M d^3s\,, \tag{2.10}$$

where using $\partial_i x_0^u = 0$, and using U to denote coordinates on the G_2-holonomy manifold, $V^U = h_{ij}\partial_i y_0^U \partial_j y^V \gamma_V \theta_0 + \frac{i}{2}\epsilon^{ijk}\partial_i y_0^U \partial_j y_0^V \partial_k y_0^W \Gamma_{VW}\theta_0$,

$$\int d\vartheta_1 d\vartheta_2 e^{\frac{i}{2l_{11}^3}\sum_{I=1}^{b_3}\sum_{\alpha=1}^{2}\sum_{i=1}^{8}(\bar{\chi}(x)\sigma^{(i)})_\alpha\vartheta_\alpha\omega_I^{(i)}} =$$
$$= -\frac{1}{4l_{11}^3}\sum_{I=1}^{b_3}\sum_{i<j=1}^{8}\omega_I^{(i)}\omega_I^{(j)}\left(\bar{\chi}\sigma^{(i)}\right)_1\left(\bar{\chi}\sigma^{(j)}\right)_2, \tag{2.11}$$

where,

$$\sigma^{(1)} \equiv \sigma^{V_2W_2}\sigma^{V_1}\,,\qquad V_{1,2}\,,W_2\in 1,2,3\,,$$
$$\omega_I^{(1)} \equiv \int_\Sigma d^3s\,\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}h_{ij}\partial_i y_0^U \partial_j y_0^{V_1}\bar{\tilde{\eta}}(y)\begin{pmatrix}-\mathbf{1}_8 & 0\\ 0 & \mathbf{1}_8\end{pmatrix}\eta(y)\,;$$

$$
\begin{aligned}
\sigma^{(2)} &\equiv \sigma^{V_2W_2}\sigma^{W_1V_1}\,, \qquad V_{1,2}\,, W_2 \in 1,2,3\,,\\
\omega_I^{(2)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}\epsilon^{ijk}\partial_i y_0^U \partial_j y_0^{V_1}\partial_k y_0^{W_1}\bar{\tilde{\eta}}(y)\gamma^{W_1V_1}\otimes \mathbf{1}_2\eta(y)\,;\\
\sigma^{(3)} &\equiv \sigma^{V_2W_2}\,, \qquad V_1 \in 4,\dots,7\,, \qquad V_2, W_2 \in 1,2,3\,,\\
\omega_I^{(3)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}h_{ij}\partial_i y_0^U \partial_j y_0^{V_1}\bar{\tilde{\eta}}(y)\begin{pmatrix} 0 & \gamma^{V_1}\\ -\gamma^{V_1} & 0\end{pmatrix}\eta(y)\,;\\
\sigma^{(4)} &\equiv \sigma^{V_2W_2}\sigma^{V_1}, \qquad V_1, W_1 \in 4,\dots,7, \qquad V_2, W_2 \in 1,2,3\,,\\
\omega_I^{(4)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}\epsilon^{ijk}\gamma^{W_1V_1}\otimes \mathbf{1}_2\partial_i y_0^U \partial_j y_0^{V_1}\partial_k y_0^{W_1}\bar{\tilde{\eta}}(y)\eta(y)\,;\\
\sigma^{(5)} &\equiv \sigma^{V_1}\,, \qquad V_1 \in 1,2,3\,, \qquad V_2, W_2 \in 4,\dots,7\,,\\
\omega_I^{(5)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}h_{ij}\partial_i y_0^U \partial_j y_0^{V_1}\bar{\tilde{\eta}}(y)\gamma^{V_2W_2}\otimes \mathbf{1}_2\begin{pmatrix} -\mathbf{1}_8 & 0\\ 0 & \mathbf{1}_8\end{pmatrix}\eta(y)\,;\\
\sigma^{(6)} &\equiv \sigma^{V_1W_1}\,, \qquad V_1, W_1 \in 1,2,3\,, \qquad V_2W_2 \in 4,\dots,7\,,\\
\omega_I^{(1)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}\epsilon^{ijk}\partial_i y_0^U \partial_j y_0^{V_1}\partial_k y_0^{W_1}\bar{\tilde{\eta}}(y)\gamma^{V_2W_2}\otimes \mathbf{1}_2\eta(y)\,;\\
\sigma^{(7)} &\equiv \mathbf{1}\,, \qquad V_{1,2}W_2 \in 4,\dots,7\,,\\
\omega_I^{(7)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}h_{ij}\partial_i y_0^U \partial_j y_0^{V_1}\bar{\tilde{\eta}}(y)\gamma^{V_2W_2}\otimes \mathbf{1}_2\begin{pmatrix} 0 & \gamma^{V_1}\\ -\gamma^{V_1} & 0\end{pmatrix}\eta(y)\,;\\
\sigma^{(8)} &\equiv \mathbf{1}, \qquad V_{1,2}, W_{1,2} \in 4,\dots,7\,,\\
\omega_I^{(1)} &\equiv \int_\Sigma d^3 s\omega_{I,UV_2W_2}(y)\sqrt{h_{ij}}\epsilon^{ijk}\partial_i y_0^U \partial_j y_0^{V_1}\partial_k y_0^{W_1}\bar{\tilde{\eta}}(y)\gamma^{V_2W_2}\otimes \mathbf{1}_2\gamma^{V_1W_1}\eta(y)\,; \qquad (2.12)
\end{aligned}
$$

where $\eta(y)$ is the seven-dimensional component of θ and $\tilde{\eta}(y)$ is the seven-dimensional component of Ψ_U. We follow the following notations for coordinates: u, v are $\mathbb{R}^4$ coordinates, $\hat{U}, \hat{V}$ are G_2-holonomy manifold coordinates that are orthogonal to the $M2$ world volume (that wraps a supersymmetric 3-cycle embedded in the G_2-holonomy manifold), and U, V are G_2-holonomy manifold coordinates. The tangent/curved space coordinates for Σ are represented by a'/m' and those for $X_{G_2}\times\mathbb{R}^4$ are represented by a''/m''.

We now come to the evaluation of $\mathcal{S}_2^\theta|_{y_0,\theta_0=0}$. Using the equality of the two $O((\delta)\Theta^2)$ terms in the action of Harvey and Moore, and arguments similar to the ones in [1], one can show that

$$
\begin{aligned}
\mathcal{S}_2^\theta|_{y_0,\theta_0=0} &= \frac{2i}{l_{11}^3}\int d^3 s\sqrt{H_{ij}}\delta\bar{\Theta}\Gamma_j D_i\delta\Theta = \frac{2i}{l_{11}^3}\int d^3 s\sqrt{h_{Ij}}h^{ij}\partial_j X^M\delta\bar{\Theta}\Gamma_M D_i\delta\Theta\\
&= \frac{2i}{l_{11}^3}\int d^3 s\sqrt{h_{ij}}h^{ij}\partial_j X^M\left[\delta\bar{\Theta}\Gamma_M\partial_i\delta\Theta + \bar{\Theta}\Gamma_M\omega_N^{AB}\Gamma_{AB}\delta\Theta\partial_i X^N\right]\\
&= \frac{2i}{l_{11}^3}\int d^3 s\sqrt{h_{ij}}\Big[h^{ij}\delta_j^{m'}e_{m'}^{a'}\delta\bar{\Theta}\left(\Gamma_{a'}\partial_i + \omega_i^{AB}\Gamma_{a'}\Gamma_{AB}\right)\delta\Theta +\\
&\qquad + h^{ij}\partial_j y'' e_{m''}^{a''}\delta\bar{\Theta}\left(\Gamma_{a''}\partial_i + \omega_i^{AB}\Gamma_{m''}\Gamma_{AB}\right)\delta\Theta\Big]\,, \qquad (2.13)
\end{aligned}
$$

from where one sees that one needs to evaluate the following bilinears: $\delta\bar{\Theta}\Gamma_{a'}\partial_i\delta\Theta$, $\delta\bar{\Theta}\Gamma_{a''}\partial_i\delta\Theta$, $\delta\bar{\Theta}\Gamma_{a'}\Gamma_{AB}\delta\Theta$, and $\delta\bar{\Theta}\Gamma_{m''}\Gamma_{AB}\delta\Theta$. One can then show (using that $\bar{\psi} = \psi^\dagger$ in euclidean space:

$$\begin{aligned}
\delta\bar{\Theta}\Gamma_3\partial_i\delta\Theta &= -\delta\theta^\dagger\sigma_3 \otimes \mathbf{1}_8\delta\Theta\,;\\
\delta\bar{\Theta}\Gamma_{a''}\partial_i\delta\Theta &= i\delta\theta^\dagger\sigma^2 \otimes \gamma_{a''}\delta\theta\,;\\
\delta\bar{\Theta}\Gamma_1\Gamma_{23}\delta\Theta &= -i\delta\theta^\dagger\sigma^3 \otimes \mathbf{1}_8\delta\theta\,;\\
\delta\bar{\Theta}\Gamma_1\Gamma_{12}\delta\Theta &= \delta\bar{\Theta}\Gamma_1\Gamma_{13}\delta\Theta = 0\,;\\
\delta\bar{\Theta}\Gamma_2\Gamma_{23}\delta\Theta &= \delta\bar{\Theta}\Gamma_2\Gamma_{23}\delta\Theta = -\delta\theta^\dagger\sigma^3 \otimes \mathbf{1}_8\delta\theta\,;\\
\delta\bar{\Theta}\Gamma_{a'}\Gamma_{b''}\delta\theta &= \delta^3_{a'}\delta\theta^\dagger\sigma^3 \otimes \gamma^{b''}\gamma^{c''}\delta\theta\,;\\
\delta\bar{\Theta}\Gamma_{a''}\Gamma_{b''c''}\delta\Theta &= -i\delta\theta^\dagger\sigma^2 \otimes \gamma^{a''}\gamma^{b''}\gamma^{c''}\delta\theta\,.
\end{aligned} \tag{2.14}$$

Using (2.14), one gets:

$$\mathcal{S}_2^\theta|_{y_0,\theta_0=0} \equiv \int_\Sigma d^3s\delta\theta^\dagger\mathcal{O}_3\delta\theta\,, \tag{2.15}$$

where

$$\begin{aligned}
\mathcal{O}_3 \equiv \frac{2i}{l_{11}^3}\sqrt{h_{ij}}\Bigg[h^{ij}\delta_j^{m'}\Bigg(&-e^3_{m'}\sigma^3 \otimes \mathbf{1}_8\partial_i - 2i\left[e^1_{m'}\omega_i^{23} + e^2_{m'}\omega_i^{31} + e^3_i\omega_i^{12}\right]\sigma^3 \otimes \mathbf{1}_8 -\\
&- 2\left[e^1_{m'}\omega_i^{13} + e^2_{m'}\omega_i^{23}\right]\sigma^3 \otimes \mathbf{1}_8 + e^3_{m'}\frac{\omega^{b''c''}}{2}\sigma^3 \otimes \gamma_{b'c'}\Bigg) +\\
&+ ih^{ij}e^{a'}_{m'}\omega_i^{b'c''}\sigma^2 \otimes \gamma_{c''}\left[\delta^{a'b'} + i\epsilon^{a'b'c'}\delta_3^{c'}\right] +\\
&+ ih^{ij}\partial_j y^{m''}e^{a''}_{m''}\left[\sigma^2 \otimes \gamma_{a''}\partial_i - \left(\omega_i^{b''c''}\sigma^2\frac{1}{6}\otimes\gamma_{a''b''c''} - \frac{1}{2}\omega^{b'c''}\delta_3^{b'}\sigma^3 \otimes \gamma_{a''c''}\right)\right]\Bigg]\,,
\end{aligned} \tag{2.16}$$

Hence, the integral over the fluctuations in θ will give a factor of $\sqrt{\det\mathcal{O}_3}$ in euclidean space.

The expression for $\mathcal{S}_2^y|_{\Sigma_{y_0,\theta_0=0}}$ is identical to the one given in [1], and will contribute $1/\sqrt{\det\mathcal{O}_1\det\mathcal{O}_2}$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are as given in the same paper:

$$\begin{aligned}
\mathcal{O}_1 &\equiv \eta_{uv}\sqrt{g}g^{ij}\mathcal{D}_i\partial_j\,,\\
\mathcal{O}_2 &\equiv \sqrt{g}(g^{ij}\mathcal{D}_i h_{\hat{U}\hat{V}}D_j + \mathcal{U}_{\hat{U}\hat{V}})\,.
\end{aligned} \tag{2.17}$$

The mass matrix $\mathcal{U}$ is expressed in terms of the curvature tensor and product of two second fundamental forms. $\mathcal{D}_i$ is a covariant derivative with indices in the corresponding spin-connection of the type $(\omega_i)^{m''}_{n''}$ and $(\omega_i)^{m'}_{n'}$, and D_i is a covariant derivative with corresponding spin connection indices of the former type.

Hence, modulo supergravity determinants, and the contribution from the fermionic zero modes, the exact form of the superpotential contribution coming from a single M2-brane wrapping an isolated supersymmetric cycle of G_2-holonomy manifold, is given by:

$$\Delta W = e^{iC - \frac{1}{l_{11}^3}\text{vol}(h)}\sqrt{\frac{\det O_3}{\det\mathcal{O}_1\ \det\mathcal{O}_2}}\,. \tag{2.18}$$

As in [6], we do not bother about 5-brane instantons, as we assume that there are no supersymmetric 6-cycles in the G_2-holonomy manifold that we consider.

3. Comparison with Harvey-Moore's paper

In [2], it is argued that for an "associative" 3-fold Σ in the G_2-holonomy manifold, the structure group Spin(8) decomposes into $\mathrm{Spin}(4)_{\mathbb{R}^4} \times \mathrm{Spin}(4)_{X_{G_2}\backslash\Sigma}$. After gauge-fixing under κ-symmetry,

$$\Theta = \left((\Theta_{--})^{AY}_{\alpha}, (\Theta_{++})^{\dot{Y}}_{\dot{\alpha}A}; 0, 0\right), \tag{3.1}$$

where $A, \overset{(\cdot)}{\alpha}, \overset{(\cdot)}{Y}$ are the $\mathrm{Spin}(3), \mathrm{Spin}(4)_{\mathbb{R}^4}, \mathrm{Spin}(4)_{X_{G_2}\backslash\Sigma}$ indices, respectively. The G_2 structure allows one to trade off $(\Theta_{--})^{AY}_{\alpha}$ for fermionic 0- and 1-forms: η, χ_i, which together with $y^u \equiv y^{\alpha\dot{\alpha}}$, form the Rozansky-Witten(RW) multiplet. Similarly, $(\Theta_{++})^{\dot{Y}}_{\dot{\alpha}A}$ gives the Mclean multiplet: $(y^{A\dot{y}}, \nu^{\dot{Y}}_{\dot{\alpha}A})$. The RW model is a $D = 3$ topological sigma model on a manifold embedded in a hyper-Kähler manifold X_{4n} [9]. If $\phi^{M(=1,\ldots,4n)}(x^i)$ are functions from mapping M to X, then the RW action is given by:

$$\int_{\Sigma} \sqrt{h_{ij}} \left[\frac{1}{2} h_{MN} \partial_i \phi^M \partial_j \phi^N h^{ij} + \epsilon_{IJ} h^{ij} \chi^I_i D_j \eta^J + \right.$$
$$\left. + \frac{1}{2\sqrt{h_{ij}}} \epsilon^{ijk} \left(\epsilon_{IJ} \chi^I_i D_j \chi^J_k + \frac{1}{3} \Omega_{IJKL} \chi^I_i \chi^J_j \chi^L_k \eta^L\right)\right], \tag{3.2}$$

where $\Omega_{IJKL} = \Omega_{JIKL} = \Omega_{IJLK}$. Then, dropping the term proportional to Ω_{IJKL}, one sees that the terms in (2.3), are very likely to give the RW action in (3.2). In [2], $n = 1$.

As the RW and Mclean's multiplets are both contained in $\delta\theta$, hence (using the notations of [2]) $\det'(L_-)\det'(\not{D}_E)$ will be given by $\det \mathcal{O}_3$ — it is however difficult to disentangle the two contributions. The relationship involving the spin connections on the tangent bundle and normal bundle (the anti self-dual part of the latter) as given in [2], can be used to reduce the number of independent components of the spin connection and thus simplify (2.17). Further, $(\det' \Delta_0)^2 |\det'(\not{D}_E)|$ should be related to $\sqrt{\det \mathcal{O}_1 \det \mathcal{O}_2}$. Hence, the order of $H_1(\Sigma, \mathbb{Z})$ must be expressible in terms of $\sqrt{\det \mathcal{O}_{1,2,3}}$ for M2-brane wrapping a rigid supersymmetric 3-cycle. However, we wish to emphasize that (2.17), unlike the corresponding result of [2], is equally valid for M2-brane wrapping a non-rigid supersymmetric 3-cycle, as considered in section 4.

4. Heat Kernel asymptotics

In this section we explore the possibility of cancellations between the bosonic and fermionic determinants.[1] For bosonic determinants $\det A_b$, the function that is relevant is $\zeta(s|A_b)$, and that for fermionic determinants $\det A_f$, the function that is additionally relevant is $\eta(s|A_f)$. The integral representation of the former involves $\mathrm{Tr}(e^{-tA_b})$, while that for the latter involves $\mathrm{Tr}(Ae^{-tA^2})$ (see [10]):

$$\zeta(s|A_b) = \frac{1}{\Gamma(2s)} \int_0^{\infty} dt t^{s-1} \,\mathrm{Tr}\left(e^{-tA_b}\right);$$

[1]We thank J.Park for suggesting to look for possible cancellations.

$$\eta(s|A_f) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dt t^{\frac{s+1}{2}}\,\mathrm{Tr}\left(A_f e^{-tA_F^2}\right), \tag{4.1}$$

where to get the UV-divergent contributions, one looks at the $t \to 0$ limit of the two terms. To be more precise (see [8]),[2]

$$\begin{aligned}
\ln\det A_b &= -\frac{d}{ds}\zeta\left(s|A_b\right)|_{s=0} = -\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\,\mathrm{Tr}\left(e^{-tA_b}\right)\right)|_{s=0};\\
\ln\det A_f &= -\frac{1}{2}\frac{d}{ds}\zeta\left(s|A_f^2\right)|_{s=0} \mp \frac{i\pi}{2}\eta\left(s|A_f\right)|_{s=0} \pm \frac{i\pi}{2}\zeta\left(s|A_f^2\right)|_{s=0}\\
&= \left[-\frac{1}{2}\frac{d}{ds} \pm \frac{i\pi}{2}\right]\left(\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\,\mathrm{Tr}\left(e^{-tA_f^2}\right)\right)|_{s=0} \mp\\
&\mp\frac{i\pi}{2}\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dt t^{\frac{s+1}{2}-1}\,\mathrm{Tr}\left(A_f e^{-tA_f^2}\right)|_{s=0},
\end{aligned} \tag{4.2}$$

where the $\mp$ sign in front of $\eta(0)$, a non-local object, represents an ambiguity in the definition of the determinant. The $\zeta(0|A_f^2)$ term can be reabsorbed into the contribution of $\zeta'(0|A_f^2)$, and hence will be dropped below. Here $\mathrm{Tr} \equiv \int dx\langle x|\ldots|x\rangle \equiv \int dx\,\mathrm{tr}(\ldots)$. The idea is that if one gets a match in the Seeley — de Witt coefficients for the bosonic and fermionic determinants, implying equality of UV-divergence, this is indicative of a possible complete cancellation.

The heat kernel expansions for the bosonic and fermionic determinants [11] are given by:

$$\begin{aligned}
\mathrm{tr}\left(e^{-tA_b}\right) &= \sum_{n=0}^\infty e_n(x, A_b)t^{(n-m)/2},\\
\mathrm{tr}\left(A_f e^{-tA_f^2}\right) &= \sum_{n=0}^\infty a_n(x, A_f)t^{(n-m-1)/2},
\end{aligned} \tag{4.3}$$

where for m is the dimensionality of the space-time. For our case, we have a compact 3-manifold, for which $e_{2p+1} = 0$ and $a_{2p} = 0$. For Laplace-type operators A_b, and Dirac-type operators A_f, the non-zero coefficients are determined to be the following:

$$\begin{aligned}
e_0(x, A_b) &= (4\pi)^{-3/2} Id,\\
e_2(x, A_b) &= (4\pi)^{-3/2}\left[\alpha_1 E + \alpha_2\tau Id\right],
\end{aligned} \tag{4.4}$$

where α_i's are constants, $\tau \equiv R_{ijji}$, and Id is the identity that figures with the scalar leading symbol in the Laplace-type operator A_b (see [11]), and

$$E \equiv B - G^{ij}\left(\partial_i\omega_j + \omega_i\omega_j - \omega_k\Gamma^k_{ij}\right), \tag{4.5}$$

where B and A_i are defined via:

$$A_b \equiv -\left(G^{ij} Id\partial_i\partial_j + A^i\partial_i + B\right), \tag{4.6}$$

[2] We are grateful to K.Kirsten for bringing [8] to our attention.

and

$$\omega_i = \frac{G_{ij}(a^j + G^{kl}\Gamma^j_{kl} Id)}{2}\,. \tag{4.7}$$

For the bosonic operators $\mathcal{O}_1$ and $\mathcal{O}_2$, the following are the relevant quantities:

$$\begin{aligned}
\mathcal{O}_1 \ &: \\
G^{ij} &\equiv -\eta_{uv}\sqrt{g}g^{ij}\,; \\
A^i &\equiv -\eta_{uv}\sqrt{g}g^{ij}\omega_j^{\mathcal{D}}\,; \\
B &= 0\,; \\
\omega_i &\equiv \omega_i^{\mathcal{D}} + \frac{\sqrt{g}g^{kl}}{2}\left[\partial_k\left(\frac{g_{li}}{\sqrt{g}}\right) + \partial_l\left(\frac{g_{ki}}{\sqrt{g}}\right) - \partial_i\left(\frac{g_{kl}}{\sqrt{g}}\right)\right], \\
\mathcal{O}_2 \ &: \\
G^{ij} &\equiv -\sqrt{g}h_{UV}g^{ij}\,; \\
A^i &\equiv -\sqrt{g}g^{ij}\left[\partial_j h_{UV} + h_{UV}\omega_j^{\mathcal{D}} + \omega_j^{\mathcal{D}}h_{UV}\right]; \\
B &\equiv -\sqrt{g}g^{ij}\left[\partial_i h_{UV}\omega_j^{D} + h_{UV}\partial_i\omega_j^{\mathcal{D}} + h_{UV}\omega_i^{D}\omega_j^{\mathcal{D}}\right] - 2\sqrt{g}\mathcal{U}_{UV}\,; \\
\omega_i &\equiv \frac{g_{ij}}{\sqrt{g}h_{UV}}\Bigg[\sqrt{g}g^{ij}\left(\partial_i h_{UV} + h_{UV}\omega_i^{D} + \omega_i^{\mathcal{D}}h_{UV}\right) + \\
&\qquad + \frac{gh_{UV}^2 g^{kl}g^{jm}}{2}\left[\partial_i\left(\frac{g_{lm}}{\sqrt{g}h_{UV}}\right) + \partial_l\left(\frac{g_{km}}{\sqrt{g}h_{UV}}\right) - \partial_m\left(\frac{g_{kl}}{\sqrt{g}h_{UV}}\right)\right]\Bigg],
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
D_i &\equiv \partial_i + \omega_i^{D}\,; \\
\mathcal{D}_i &\equiv \partial_i + \omega_i^{\mathcal{D}}\,.
\end{aligned} \tag{4.9}$$

To actually evaluate e_0 and e_2, we need to find an example of a regular G_2-holonomy manifold that is metrically $\Sigma \times M_4$, where Σ is a supersymmetric 3-cycle on which we wrap an M2-brane once, and M_4 is a four manifold. One such example was obtained in [13], which is a G_2-holonomy manifold that is $M_4 \times T^3$, metrically. It is given below:

$$ds_7^2 = dr^2 + \frac{r^2}{4}\sum_{i=1}^{3}\sigma_i^2 + \sum_{i=1}^{3}\alpha_i^2\,, \tag{4.10}$$

where σ_i's are left-invariant one-forms obeying the SU(2) algebra: $d\sigma_i = -\frac{1}{2}\epsilon^{ijk}d\sigma^j \wedge d\sigma^k$, given by:

$$\begin{aligned}
\sigma_1 &\equiv \cos\psi d\theta + \sin\psi\sin\theta d\phi\,, \\
\sigma_2 &\equiv -\sin\psi d\theta + \cos\psi\sin\theta d\phi\,, \\
\sigma_3 &\equiv d\psi + \cos\theta d\phi\,,
\end{aligned} \tag{4.11}$$

and α_i's are harmonic one-forms that form a basis for $H^1(T^3, \mathbb{R})$. One can write $\alpha_i = d\theta_i$. To see that the T^3 corresponds to a supersymmetric 3-cycle, we need to show that the

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pull-back of the calibration Φ_3 restricted to Σ, is the volume form on Σ (see [14]). Φ_3 using the notations of [13] is given by:

$$\Phi_3 = e^{125} + e^{147} + e^{156} - e^{246} + e^{237} + e^{345} + e^{567}\,, \tag{4.12}$$

where $e^{ijk} \equiv e^i \wedge e^j \wedge e^k$. Let $1, \ldots, 7$ denote $r, \psi, \theta, \phi, \theta_1, \theta_2, \theta_3$. Hence, when restricted to $\Sigma(\theta_1, \theta_2, \theta_3)$ using the static gauge, one gets:

$$\Phi_3|_\Sigma = e^{567} = d\theta_1 \wedge d\theta_2 \wedge d\theta_3\,, \tag{4.13}$$

which is the volume form on T^3. Thus, the T^3 of (4.10) is a supersymmetric 3-cycle.

For (4.10), one sees that $g_{ij} = \delta_{ij} + \partial_i y_0^{\hat{U}} \partial_j y_0^{\hat{V}} g_{\hat{U}\hat{V}}$, having used the definition of g_{ij} as a pull-back of the space-time metric g_{MN}, static gauge and that $\partial_i y_0^u = 0$. If one assumes that the coordinates r, ψ, θ, ϕ are very slowly varying functions of $\theta_1, \theta_2, \theta_3$, one sees that $g_{ij} \sim \delta_{ij}$. This simplifies the algebra, though one can work to any desired order in $(\partial_i y_0^{\hat{U}})^{p(>0)}$, and get conclusions similar to the ones obtained below.

Using the reasoning as given in appendix A, one gets:

$$\begin{aligned} e_0(x, \mathcal{O}_1) &= (4\pi)^{-3/2}\,; \\ e_2(x, \mathcal{O}_1) &= 0\,, \end{aligned} \tag{4.14}$$

and:

$$\begin{aligned} e_0(x, \mathcal{O}_2) &= (4\pi)^{-3/2}\,; \\ e_2(x, \mathcal{O}_2) &= (4\pi)^{-3/2} \frac{72\alpha_2}{r^4}\,. \end{aligned} \tag{4.15}$$

We now do a heat-kernel asymptotics analysis of the fermionic determinant $\det \mathcal{O}_3$. The fermionic operator $\mathcal{O}_3$ can be expressed as:

$$\mathcal{O}_3 \equiv \sqrt{h} h^{ij} \Gamma_j D_i = \sqrt{h} h^{ij} \Gamma_j \left(\partial_i + 2\frac{1}{4} \omega_i^{a'b'} \Gamma_{a'b'} + \frac{1}{4} \omega_i^{a'b''} \Gamma_{a'b''} \right) \equiv G^{ij} \Gamma_j \partial_i - r\,, \tag{4.16}$$

where

$$\begin{aligned} G^{ij} &\equiv \sqrt{h} h^{ij}\,; \\ r &\equiv \frac{-1}{4} \sqrt{h} h^{ij} \Gamma_j \left(\omega_i^{a'b'} \Gamma_{a'b'} + \omega_i^{a'b''} \Gamma_{a'b''} \right). \end{aligned} \tag{4.17}$$

$\mathcal{O}_3$ is of the Dirac-type as $\mathcal{O}_3^2$ is of the Laplace-type, as can be seen from the following:

$$\begin{aligned} \mathcal{O}_3^2 &\equiv G^{ij} \partial_i \partial_j + A^i \partial_i + B\,, \qquad \text{where :} \\ G^{ij} &\equiv h h^{ij}\,; \\ A^i &\equiv \sqrt{h} \Gamma^j \Gamma_k \partial_j \left(\sqrt{h} h^{kl} \right) + 2h \Gamma^i \Gamma^l \omega_l^{CD} \Gamma_{CD}\,; \\ B &\equiv \sqrt{h} \Gamma^j \Gamma_k \Gamma_{CD} \partial_j \left(\sqrt{h} h^{kl} \omega_l^{CD} \right) + h \Gamma^j \Gamma^l \omega_j^{AB} \Gamma_{AB} \omega_l^{CD} \Gamma_{CD}\,. \end{aligned} \tag{4.18}$$

Now,

$$\mathcal{O}_3 \equiv G^{ij} \Gamma_j \bigtriangledown_i - \phi\,, \tag{4.19}$$

where

$$\phi \equiv r + \Gamma^i \omega_i \,, \tag{4.20}$$

and

$$\begin{aligned}\omega_l \equiv{}& \frac{G_{il}}{2}(-\Gamma^j \partial_j \Gamma^i + \{r, \Gamma^i\} + G^{jk}\Gamma^i_{jk}) = \\ ={}& \frac{h_{il}}{2\sqrt{h}}\left(\frac{1}{4}\sqrt{h} h^{i'j'}\left\{\Gamma_{j'}\left(\omega_{i'}^{a'b'}\Gamma_{a'b'} + \omega_{i'}^{a'b''}\Gamma_{a'b''}\right), \Gamma^i\right\} + \right. \\ & \left. + \frac{h h^{jk} h^{ii'}}{2}\left(\partial_j\left[\frac{h_{ki'}}{\sqrt{h}}\right] + \partial_k\left[\frac{h_{ji'}}{\sqrt{h}}\right] - \partial_{i'}\left[\frac{h_{jk}}{\sqrt{h}}\right]\right)\right).\end{aligned} \tag{4.21}$$

The Seeley-de Witt coefficients a_i are given by (see [12]):

$$\begin{aligned} a_1(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) &= -(4\pi)^{-3/2}\,\mathrm{tr}(\phi)\,; \\ a_3(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) &= -\frac{1}{6}(4\pi)^{-3/2}\,\mathrm{tr}(\phi\tau + 6\phi\mathcal{E} - \Omega_{a'b';a'}\Gamma_{b'})\,, \end{aligned} \tag{4.22}$$

where

$$\mathcal{E} \equiv -\frac{1}{2}\Gamma^i\Gamma^j\Omega_{ij} + \Gamma^i\phi_{;i} - \phi^2\,, \tag{4.23}$$

and

$$\Omega_{ij} \equiv \partial_i\omega_j - \partial_j\omega_i + [\omega_i, \omega_j]\,. \tag{4.24}$$

Now, e.g. $\Gamma^i = \partial^i y^M \Gamma_M$, where $y^M \equiv y^{m', \hat{U}, u}$ and given that $\partial_i y^u = 0$, then in the static gauge, $\Gamma^i = \delta^i_{m'}\Gamma_{m'} + \partial^i y^{\hat{U}}\Gamma_{\hat{U}} = \delta^i_{m'} e_{m'}^{\ a'}\Gamma_{a'} + \partial^i y^{\hat{U}} e_{\hat{U}}^{\ \hat{A}}\Gamma_{\hat{A}}$. Now, lets make the simplifying assumption as done for the bosonic operators, we assume that $y^{\hat{U}}$ varies very slowly w.r.t. the M2-brane world-volume coordinates. Hence, we drop all terms of the type $(\partial_i y^{\hat{U}})^{p(>0)}$. The conclusion below regarding the vanishing of the Seeley-de Witt coefficients a_1 and a_3, will still be valid by generalizing (4.27). The dropping of $(\partial_i y^U)^{p(>0)}$-type terms will be indicated by $\sim$ as opposed to $=$ in the equations below.

Let (m, n) denote the number of ($\Gamma_{a'}$'s, $\Gamma_{a''}$'s). Then, without taking into account the multiplicity of the different types of terms,

$$\begin{aligned} \phi &\sim \text{(odd,even)} + \text{(even,odd)}\,; \\ \omega_i &\sim \text{(even,even)} + \text{(odd,odd)}\,; \\ \phi_{;i} &\sim \text{(odd,even)} + \text{(even,odd)}\,, \end{aligned} \tag{4.25}$$

where; denotes covariant differentiation, using which one gets

$$\begin{aligned} \phi^3 &\sim \text{(odd,even)} + \text{(even,odd)}\,; \\ \phi\Gamma^i\Gamma^j\Omega_{ij} &\sim \text{(odd,even)} + \text{(even,odd)}\,; \\ \Omega_{a'b';b'}\Gamma_{a'} &\sim \text{(odd,even)} + \text{(even,odd)}\,. \end{aligned} \tag{4.26}$$

Using (2.9), one sees that

$$\operatorname{tr}\left(\prod_{i=1}^{2m+1}\Gamma_{a'_i}\right) = \operatorname{tr}\left(\prod_{i=1}^{2m}\Gamma_{a'_i}\Gamma_{a''}\right) = \operatorname{tr}\left(\prod_{i=1}^{2m+1}\Gamma_{a'_i}\Gamma_{a''b''}\right)$$
$$= \operatorname{tr}\left(\prod_{i=1}^{2m}\Gamma_{a'_i}\Gamma_{a''}\Gamma_{b''}\Gamma_{c''}\right) = 0\,. \tag{4.27}$$

Hence,

$$a_1(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) = a_3(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) \sim 0\,. \tag{4.28}$$

We conjecture that in fact, $a_{2n+1}(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) \sim 0$ for $n = 0, 1, 2, 3, \ldots$.

By using reasoning similar to the one used in appendix A, one can show that:

$$\begin{aligned} e_0\left(x, \mathcal{O}_3^2\right) &= (4\pi)^{-3/2}\,; \\ e_2\left(x, \mathcal{O}_3^2\right) &\sim 0\,. \end{aligned} \tag{4.29}$$

From the extra factor of $1/2$ multiplying the $\zeta'(0|\mathcal{O}_3^2)$ relative to $\zeta'(0|\mathcal{O}_1)$ in (4.2) and (4.28) and (4.29), one sees the possibility that:

$$\frac{\det \mathcal{O}_3}{\det \mathcal{O}_1} \sim \frac{1}{2}\,. \tag{4.30}$$

In conclusion, one sees that Seeley-de Witt coefficients of the fermionic operator $\mathcal{O}_3$ are proportional to those of the bosonic operator $\mathcal{O}_1$ in the adiabatic approximation, to the order calculated, for the G_2-metric (4.10). This is indicative of possible cancellation between them. This is expected, as the M2-brane action has some supersymmetry. As $b_1(T^3) = 3 > 0$, thus the supersymmetric 3-cycle of (4.10) is an example of a non-rigid supersymmetric 3-cycle. The result of [2] is not applicable for this case. On the other hand, the superpotential written out as determinants, as in this work, is still valid. The corresponding modified formula in [2] might consist, as prefactors, in addition to the phase, the torsion elements of $H_1(\Sigma, \mathbb{Z})$, represented by $|H_1(\Sigma, \mathbb{Z})|'$ in [9], and perhaps a geometrical quantity that would encode the G_2-analog of the arithmetic genus condition in the context of 5-brane instantons obtained by wrapping M5-brane on supersymmetric 6-cycles in CY_4 in [21].[3] The validity of the arithmetic genus argument, even for CY_4, needs to be independently verified though [15]. The above cancellation implies that for the G_2-metric (4.10), the geometric prefactors will be $\sim \sqrt{1/2 \det \mathcal{O}_2}$. This could perhaps be a generic feature with all G_2-metrics that are metrically $\Sigma \times M_4$.

5. Conclusion and discussions

In this paper, we have evaluated in a closed form, the exact expression for the non-perturbative contribution to the superpotential from a single M2-brane wrapping an isolated supersymmetric 3-cycle of a G_2-holonomy manifold. The comparison with Harvey

[3] We are grateful to S.Gukov for telling us about the topological condition involving the arithmetic genus in [21].

and Moore's result, is suggestive but not exact. A heat-kernel asymptotics analysis for a non-compact smooth G_2-holonomy manifold that is metrically $\mathbb{R}^4 \times T^3$, in the adiabatic approximation, showed that the UV-divergent terms of one of the bosonic and the fermionic determinants are proportional to each other, to the order we calculate, indicative of cancellation between the same, as expected because the M2-brane action of Bergshoeff, Sezgin and Townsend is supersymmetric. *Unlike the result of [2], the expression obtained for the superpotential above in terms of fermionic and bosonic determinants, in addition to a holomorphic phase factor, is valid even for non-rigid supersymmetric 3-cycles as the one considered in (4.10) above.*

Following [6], it is tempting to conjecture that the superpotential term corresponding to multiple wrappings of the M2-brane around the supersymmetric 3-cycle, should be give by:

$$\Delta W = \sum_n \sqrt{\frac{\det \mathcal{O}_3}{\det \mathcal{O}_1 \det \mathcal{O}_2}} \frac{e^{n\int_\Sigma [iC - \frac{1}{l_{11}^3}\mathrm{vol}(h)]}}{n^2}. \tag{5.1}$$

The calculations in this paper were done for a smooth G_2-holonomy manifold. However, for singular G_2-holonomy manifolds with ADE-type singularity, e.g. spin bundles over S^3 with the topology $\frac{\mathbb{C}^2}{\Gamma_{ADE}} \times S^3$, one can explore the following in more details. Based on [19], wherein the four-dimensional superpotential of Super Yang-Mills theory, is calculated by compactifying the theory on a circle, and then taking its four-dimensional limit, it was suggested in [20] that the non-perturbative superpotential of membrane instantons in M-theory on $\frac{\mathbb{C}^2}{\Gamma_{ADE}} \times S^3 \times S^1 \times \mathbb{R}^{2,1}$ can be evaluated by compactifying M-theory on the S^1 to type-IIA theory on $\frac{\mathbb{C}^2}{\Gamma_{ADE}} \times S^3 \times \mathbb{R}^{2,1}$ and evaluating the superpotential from the wrapping of D2-branes on the (supersymmetric) S^3, whose world-volume theory in the large volume limit of the S^3, is given by a "quiver gauge theory" [18]. The wrapped D2-branes are constrained to live at the singularities (that lie at the origin of $\mathbb{C}^2/\Gamma_{ADE}$) of the G_2-holonomy manifold, which in the quiver gauge theory, is the locus in the moduli space, where the D2-branes fractionate. These fractional D2-branes are supposed to be D4-branes wrapping vanishing 2-cycles at the origin in $\mathbb{C}^2/\Gamma_{ADE}$.

In terms of relating the result obtained in (2.18) to that of the 1-loop Schwinger computation of M-theory and the large-N limit of the partition function evaluated in [16],[4] one notes that the 1-loop Schwinger computation also has as its starting point, an infinite-dimensional bosonic determinant of the type $\det((i\partial - eA)^2 - Z^2)$, A being the gauge field corresponding to an external self-dual field strength, and Z denoting the central charge. The large-N limit of the partition function of Chern Simons theory on an S^3, as first given by Periwal in [17], involves the product of infinite number of sin's, that can be treated as the eigenvalues of an infinite determinant. This is indicative of a possible connection between the membrane instanton contribution to the superpotential, the 1-loop Schwinger computation and the large-N limit of the Chern-Simons theory on an S^3.

[4]This logic was suggested to us by R.Gopakumar.

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A. Seeley-de Witt coefficients for bosonic determinants in the superpotential

In this appendix we derive (4.14) and (4.15).

Lets first consider the Seeley de-Wit coefficients for $\mathcal{O}_1$. Now, in the above adiabatic approximation, the world volume metric of the M2-brane is flat. Hence, the Christoffel connection Γ^i_{jk} in (4.7) for $\mathcal{O}_1$, vanishes. Now, $\omega_i^{a'b'} \sim \delta_i^{m'}\omega_{m'}^{a'b'}$, where

$$\omega^{ab}_{m'} = e^{[a}{}_{n'} g^{n'l'} (\partial_{m'} e_{l'}{}^{b']} - \Gamma^{p'}_{l'm'} e_{p'}{}^{b']})\,, \tag{A.1}$$

the antisymmetry indicated on the right hand side of (A.1) being applicable only to the tangent-space indices a', b', and where for (4.10), the following are the non-zero vielbeins:

$$\begin{aligned}
e^1{}_r &= 1\,;\\
e^2{}_\theta &= \frac{r}{2}\cos\psi\,, \qquad e^2{}_\phi = \frac{r}{2}\sin\psi\sin\theta\,;\\
e^3{}_\theta &= -\frac{r}{2}\sin\psi\,, \qquad e^3{}_\phi = \cos\psi\sin\theta\,;\\
e^4{}_\psi &= \frac{r}{2}\,, \qquad e^4{}_\phi = \frac{r}{2}\cos\theta\,;\\
e^5{}_{\theta_1} &= e^6{}_{\theta_2} = e^7{}_{\theta_3} = 1\,.
\end{aligned} \tag{A.2}$$

Hence, for the G_2 metric of (4.10), $\partial_{m'} e_{l'}{}^{b'} = 0$. Also, $\Gamma^{p'}_{l'm'} = 0$. Thus, $\omega_i^{\mathcal{D}} \sim 0$.

In the adiabatic approximation, $\tau \sim 0$.

Hence,

$$\begin{aligned}
e_0(x, \mathcal{O}_1) &= (4\pi)^{-3/2}\,;\\
e_2(x, \mathcal{O}_1) &= 0\,.
\end{aligned} \tag{A.3}$$

We next consider evaluation of $e_{0,2}(x, \mathcal{O}_2)$. Once again, the Christoffel connection Γ^i_{jk} vanishes. Again, $\omega^{D,\mathcal{D}} \sim 0$. Also, $\partial_i h_{\hat{U}\hat{V}} \sim 0$. Hence, $A^i \sim 0$ that figures in (4.6).

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Now,

$$\mathcal{U}_{\hat{U}\hat{V}} \equiv \frac{1}{2} R^{m'}_{\hat{U}m'\hat{V}} + \frac{1}{8} Q^{m'n'}_{\hat{U}} Q_{m'n'\hat{V}}\,, \tag{A.4}$$

where the second fundamental form is defined via:

$$\Gamma^{m''}_{k'l'} \equiv -\frac{1}{2} g^{m''n''} Q_{k'l'n''}\,. \tag{A.5}$$

Using:

$$R^{m'}_{\hat{U}m'\hat{V}} = \partial_{\hat{V}} \Gamma^{m'}_{\hat{U}m'} - \partial_{m'} \Gamma^{m'}_{\hat{U}\hat{V}} + \Gamma^{V}_{\hat{U}m'} \Gamma^{m'}_{V\hat{V}} - \Gamma^{V}_{\hat{U}\hat{V}} \Gamma^{m'}_{Vm'}\,, \tag{A.6}$$

and the fact that the non-zero Christoffel symbols do not involve m' as one of the (three) indices and that their values are m'-independent, one sees that

$$\mathcal{U}_{\hat{U}\hat{V}} = 0\,. \tag{A.7}$$

Hence, $B \sim 0$ that figures in (4.6).

For evaluating $\tau \equiv g^{i_1 i_2} g^{j_1 j_2} R_{i_1 j_1 j_2 i_2} = g^{i_1 i_2} g^{j_1 j_2} g_{i_1 l_1} R^{l_1}_{j_1 j_2 i_2}$, one needs to evaluate $R^{l_1}_{j_1 j_2 i_2} = \partial_{i_2} \Gamma^{l_1}_{j_1 j_2}$ -$\partial_{j_2} \Gamma^{l_1}_{j_1 i_2} + \Gamma^{p}_{j_1 j_2} \Gamma^{l}_{p i_2}$ -$\Gamma^{p}_{j_1 i_2} \Gamma^{l_1}_{p j_2}$. This will be evaluated using the metric given by $G^{ij} = g^{ij} \sqrt{g} h_{\hat{U}\hat{V}}$, as given in (4.9), where we will use the adiabatic approximation: $g_{ij} \sim \delta^{m'}_{i} \delta^{n'}_{j} g_{m'n'}$. Due to the $\hat{U}\hat{V}$ indices, the Ricci scalar τ is actually a matrix in the $X_{G_2} \setminus \Sigma$ space. In the adiabatic approximation, only the product of the two Christoffel symbols in $R^{l_1}_{j_1 j_2 i_2}$ is non zero, and is given by the following expression:

$$\begin{aligned}
&\Gamma^{p''}_{j_1 j_2} \Gamma^{l_1}_{p'' i_2} - \Gamma^{p''}_{j_2 i_1} \Gamma^{l_1}_{p'' j_2} = \\
&= -\frac{h_{\hat{U}\hat{V}}}{4} \delta_{j_1 j_2} \delta^{l_1}_{i_2} \left[\left(\partial_r \left[\frac{1}{h_{\hat{U}\hat{V}}} \right] \right)^2 + \frac{4}{r^2} \left(\partial_\theta \left[\frac{1}{h_{\hat{U}\hat{V}}} \right] \right)^2 \right] \\
&= -\frac{\delta_{j_1 j_2} \delta^{l_1}_{i_2}}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{16}{r^4} & 0 & \frac{16}{r^4 \cos\theta} + \frac{4 \sin^2\theta}{r^2 \cos^3\theta} \\ 0 & 0 & \frac{16}{r^4} & 0 \\ 0 & \frac{16}{r^4 \cos\theta} + \frac{4 \sin^2\theta}{r^2 \cos^3\theta} & 0 & \frac{16}{r^4} \end{pmatrix}.
\end{aligned} \tag{A.8}$$

Hence, on taking $tr_{X_{G_2} \setminus \Sigma}$, one gets:

$$\tau = \frac{72}{r^4}\,. \tag{A.9}$$

Hence,

$$\begin{aligned}
e_0(x, \mathcal{O}_2) &= (4\pi)^{-3/2}\,; \\
e_2(x, \mathcal{O}_2) &= (4\pi)^{-3/2} \frac{72 \alpha_2}{r^4}\,.
\end{aligned} \tag{A.10}$$

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